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Downward Sets and their separation and approximation properties

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Abstract. We develop a theory of downward subsets of the space $\mathbb{R}^I$, where I is a finite index set. Downward sets arise as the set of all solutions of a system of inequalities $x \in \mathbb{R}^I,\ f_t(x) \leqslant 0\ (t \in T)$, where T is an arbitrary index set and each $f_t\ (t \in T)$ is an increasing function defined on $\mathbb{R}^I$. These sets play an important role in some parts of mathematical economics and game theory. We examine some functions related to a downward set (the distance to this set and the plus-Minkowski gauge of this set, which we introduce here) and study lattices of closed downward sets and of corresponding distance functions. We discuss two kinds of duality for downward sets, based on multiplicative and additive min-type functions, respectively, and corresponding separation properties, and we give some characterizations of best approximations by downward sets. Some links between the multiplicative and additive cases are established.

Key words: Abstract convex function; Abstract convex set; Downward set; Min-type coupling function; Plus-Minkowski gauge

1. Introduction

It is well known that there exist various kinds of necessary and sufficient conditions for the global minimum of a convex function over a convex set. These conditions play a fundamental role in convex optimization. However, it is very difficult to obtain verifiable conditions for a global minimum in general non-convex optimization. Thus it is important to describe some classes of non-convex global optimization problems, where such conditions can be obtained. In particular, one can examine some classes of multi-extremal problems, having a certain structure which is different from convexity, such as monotonic structure.

One of the well known problems of convex optimization is that of best approximation by elements of convex sets. Convexity can be also used for best approximation by complements of convex sets. Best approximation by different kinds of sets is a very complicated problem. Monotonic structure was used in [13] for establishing necessary and sufficient conditions for best approximation by elements of the so-called normal subsets of the cone $\mathbb{R}^n_+$. In this paper we study a new class of multi-extremal problems of best approximation in $\mathbb{R}^n$ with monotonic data. First we develop special techniques for examining such problems. Such techniques are useful in the study of some problems related to inequalities defined by increasing functions and some problems of game theory. We believe that these techniques will be useful in the study of various global optimization problems with monotonic data.

Let $I = \{1, ..., n\}$ and $\mathbb{R} = (-\infty, +\infty)$, the real line. Denote by $\mathbb{R}^I$ the space of all vectors $x = (x_i)_{i\in I}$, endowed with the max-norm and the coordinatewise order relation. In this paper we shall study downward subsets of the space $\mathbb{R}^I$, that is, sets $A \subseteq \mathbb{R}^I$ such that $(x \in A,\ y \leqslant x) \;\Rightarrow\; y \in A$. We use the notation $\mathbb{R}^I$ rather than $\mathbb{R}^n$, since some of our results remain valid, with the same proofs, for bounded functions on an arbitrary index set I (mutatis mutandis, e.g., replacing $\min_{i\in I}$ by $\inf_{i\in I}$ in formula (1.3) below).

Convex downward subsets of $\mathbb{R}^I$ play an important role in some parts of mathematical economics and cooperative game theory, where they are called *comprehensive* sets. Not necessarily convex downward sets have already found applications in the theory of games with non-transferable utility (see, for example [10]). We hope that the theory developed in this paper can be used to extend some results of mathematical economics and cooperative game theory.

Downward sets arise as the set of all solutions of a system of inequalities

$$x \in \mathbb{R}^I,\ f_t(x) \leqslant 0 \qquad (t \in T), \tag{1.1}$$

where T is an arbitrary index set and, for each $t \in T,\ f_t$ is an increasing function defined on $\mathbb{R}^I$. We shall give a representation of downward sets by means of special increasing functions.

Downward sets can be considered as a certain analogue of normal subsets of the cone $\mathbb{R}^I_+$. By definition, a set $G \subseteq \mathbb{R}^I_+$ is normal if $(x \in G,\ x' \in \mathbb{R}^I_+,\ x' \leqslant x) \;\Rightarrow\; x' \in G$. Normal sets have been studied in [11, 13]. This study has been based on the application of the 'multiplicative coupling function' $\varphi_0 : \mathbb{R}^I_+ \times \mathbb{R}^I_+ \to \mathbb{R}_+$ defined (see, e.g., [11]) by the 'scalar product'

$$\varphi_0(x,l) = \langle l, x\rangle := \begin{cases} \min_{i\in I(l)} l_i x_i & \text{if } x \in \mathbb{R}^I_+, l \in \mathbb{R}^I_+\backslash\{0\} \\ 0 \quad \text{if } x \in \mathbb{R}^I_+, l = 0, \end{cases} \tag{1.2}$$

where $I(l) := \{i \in I : l_i \neq 0\}$, and of the 'min-type functions' (more precisely, multiplicative min-type functions), i.e., functions of the form $\varphi_0(., l)$, with $l \in \mathbb{R}^I_+$ and φ_0 of (1.2). Among other results, in [13] it has been shown that a subset A of $\mathbb{R}^I$ is closed and normal if and only if it is abstract convex with respect to the set of all multiplicative min-type functions, i.e., if and only if A and any outside point can be separated by a multiplicative min-type function. However, in contrast with the case of normal sets, multiplicative min-type functions sometimes are not very convenient in the study of downward sets. We shall show that the 'additive coupling function' $\varphi : \mathbb{R}^I \times \mathbb{R}^I \to \mathbb{R}$ defined (see e.g. [3, 4] and References therein) by the 'scalar product'

$$\varphi(x,l) := \min_{i\in I}(l_i + x_i) \quad (x \in \mathbb{R}^I, l \in \mathbb{R}^I), \tag{1.3}$$

and the 'additive min-type functions', i.e., functions of the form $\varphi(.,l)$, with $l \in \mathbb{R}^I$ and φ of (1.3), are more suitable for this purpose. We shall prove that a subset A of $\mathbb{R}^I$ is closed and downward if and only if it is abstract convex with respect to the set of all additive min-type functions, i.e., if and only if A and any outside point can be separated by an additive min-type function. Also, we shall demonstrate that downward sets, through the notion of downward hull, form a useful tool in the study of normal sets.

Another area of applications of downward sets is the theory of best approximation. The theory of best approximation by elements of convex and reverse convex sets (that is, complements of convex sets) is well-developed and has found applications in many areas of mathematics. However, convexity is sometimes a very restrictive assumption, so there is a clear need to study the best approximation by not necessarily convex sets. In this direction, the theory of best approximation in $\mathbb{R}^I_+$ endowed with the max-norm, by elements of normal sets, has been developed in [13]. In the present paper we shall study best approximation in $\mathbb{R}^I$ by elements of downward sets.

For the theory of best approximation in $\mathbb{R}^I_+$ endowed with the max-norm, by elements of normal sets, in [13] it has been necessary to study the more difficult problem of the separation of a normal set and a certain ball (in the max-norm). While in [13] it has been shown that the separability of a normal set and a ball by multiplicative min-type functions can be used to obtain necessary and sufficient conditions *for the least* element of best approximation by a normal subset of $\mathbb{R}^I_+$, we shall show that in the case of downward sets in $\mathbb{R}^I$ the situation is better, namely, the separability of a downward set and a ball by multiplicative or additive min-type functions can be used to establish necessary and sufficient conditions *for any* element of best approximation by a downward subset of $\mathbb{R}^I$.

Moreover, we shall study the expression of the distance from a given point x to a downward set A, as well as the distance to the set A as a function of a variable point x. We shall describe properties of this function and establish its links with the so-called plus-Minkowski gauge. We shall study not only individual downward sets and distance functions, but also the lattice of all closed downward sets and the corresponding lattice of all distance functions.

The above mentioned results will allow us to extend to closed downward sets the linear regularity result obtained in [12] for closed normal sets. Recall that a collection $(A_t)_{t\in T}$ of subsets of a normed linear space X is called *linearly regular* if there exists a constant $C > 0$ such that for all $x \in X$

$$\operatorname{dist}(x, \bigcap_{t\in T} A_t) \leqslant C \sup_{t\in T} \operatorname{dist}(x, A_t). \tag{1.4}$$

This concept plays an important role in the theory of error bounds for convex inequality systems. Among other results, it is known (see [1], Corollary 5.26) that if the collection $\{A_t\}_{t\in T}$ is finite and each A_t is a convex polyhedron, then $\{A_t\}_{t\in T}$ is linearly regular, and in [2], fact 2.15, it has been observed that this is a reformulation of the classical Hoffman error bound result [5].

It was shown in [12] that for closed normal subsets of $\mathbb{R}^I_+$ the situation is completely different, namely an arbitrary collection $(A_t)_{t\in T}$ of closed normal sets is linearly regular with $C = 1$, and for all $x \in \mathbb{R}^I_+$ we have even

$$\operatorname{dist}(x, \bigcap_{t\in T} A_t) = \sup_{t\in T} \operatorname{dist}(x, A_t). \tag{1.5}$$

We shall demonstrate that (1.5) holds also for an arbitrary collection $(A_t)_{t\in T}$ of closed downward sets in $\mathbb{R}^I$ and all $x \in \mathbb{R}^I$.

The structure of the paper is as follows. In Section 2 we present some preliminary results. In Section 3 we discuss some properties of the distance to a downward set, introduce and study the plus-Minkowski gauge of a downward set and establish some links between the distance and the plus-Minkowski gauge. Connections between downward subsets of $\mathbb{R}^I$ and normal subsets of the cone $\mathbb{R}^I_+$, based on the notion of the downward hull of a normal set, are studied in Section 4. The complete lattices of all closed downward sets, all closed normal sets and all distance functions, and connections between these lattices, are examined in Section 5. In the rest of the paper we study two types of duality, their applications, and relations between them. Dualities based on multiplicative and additive min-type functions and their application to separation of a closed downward set and a ball are discussed in Sections 6 and 7, respectively. Applications of these separation properties to characterizations of best approximations by downward sets can be found in Section 8. A representation of the distance to a downward set through distances to lower min-type half-spaces is given in Section 9. In the final Section 10 we discuss some connections between the multiplicative and additive cases.

2. Preliminaries

Let I be a finite set of indices. Consider the space $\mathbb{R}^I$ of all vectors $(x_i)_{i\in I}$. We shall use the following notations:

- if $x \in \mathbb{R}^I$, then x_i is the i-th coordinate of x;
- if $x, y \in \mathbb{R}^I$ then $x \geqslant y \iff x_i \geqslant y_i$ for all $i \in I$;
- if $x, y \in \mathbb{R}^I$ then $x \gg y \iff x_i > y_i$ for all $i \in I$;
- $\mathbb{R}^I_+ = \{x = (x_i)_{i\in I} \in \mathbb{R}^I : x_i \geqslant 0 \text{ for all } i \in I\}$;
- $\mathbb{R}^I_{++} = \{x = (x_i)_{i\in I} \in R^I : x_i > 0 \text{ for all } i \in I\}$;
- $\mathbf{1} = (1, \dots, 1)$;
- for each $x \in \mathbb{R}^I$, $x^+ = \max(x, 0)$ (coordinatewise, i.e., $x_i^+ = \max(x_i, 0)$ for all $i \in I$).

In the sequel we shall asume that the space $\mathbb{R}^I$ is equipped with the coordinatewise order relation $\geqslant$ and with the max-norm $\|x\| = \|x\|_\infty := \max_{i\in I} |x_i|$. Note that the ball $B(x, r) := \{y : \|y - x\| \leqslant r\}$ can be represented in the form $B(x, r) = \{y : x - r\mathbf{1} \leqslant y \leqslant x + r\mathbf{1}\}$. For any set $A \subseteq \mathbb{R}^I$, we shall denote by int A, cl A, and bd A, the interior, the closure and the boundary of A, respectively. If there exists the least element of A, we shall denote it by min A.

DEFINITION 1. A nonempty subset A of the space $\mathbb{R}^I$ is called *downward* if $(x \in A, x' \leqslant x) \;\Rightarrow\; x' \in A$. We accept, by definition, that *the empty set* $\emptyset$ *is downward.*

A simple example of a downward set is a set of the form $\{y \in \mathbb{R}^I : y \leqslant x\}$, where $x \in \mathbb{R}^I$. It is easy to check that a function $f : \mathbb{R}^I \to \bar{R}$ is increasing (i.e., $(x, y \in \mathbb{R}^I, x \leqslant y) \Rightarrow f(x) \leqslant f(y))$ if and only if its level sets $S_c(f) := \{y : f(y) \leqslant c\}$ are downward for all $c \in \mathbb{R}$.

PROPOSITION 1. *Let* $A \subset \mathbb{R}^I$ *be a downward set and* $x \in \mathbb{R}^I$.

(a) *If* $x \in A$, *then* $x - \varepsilon\mathbf{1} \in$ int A *for all* $\varepsilon > 0$. *Hence,* int $A \neq \emptyset$.

(b) *We have*

$$\text{int } A = \left\{x \in \mathbb{R}^I : \exists \varepsilon > 0,\ x + \varepsilon\mathbf{1} \in A\right\}. \tag{2.6}$$

(c) *If A is closed, then it is regular, that is,* $A =$cl int A.

Proof. (a) Let $x \in A$ and $\varepsilon > 0$. Since the greatest element of the ball $B := B(x - \varepsilon\mathbf{1}, \varepsilon)$ is $(x - \varepsilon\mathbf{1}) + \varepsilon\mathbf{1} = x \in A$, and since A is downward, it follows that $B \subset A$. Hence, $x - \varepsilon\mathbf{1} \in$int A, so int $A \neq \emptyset$.

(b) If there exists $\varepsilon > 0$ such that $x + \varepsilon\mathbf{1} \in A$, then, by (a), we have $x = (x + \varepsilon\mathbf{1}) - \varepsilon\mathbf{1} \in$int A. Conversely, if $x \in$int A, then there exists a ball $B(x, \varepsilon) \subset A$ (with $\varepsilon > 0$). Since $x + \varepsilon\mathbf{1} \in B(x, \varepsilon)$, it follows that $x + \varepsilon\mathbf{1} \in A$.

(c) Taking $\varepsilon \to 0$ in part (a), it follows that $x \in$cl int A. Thus, $A \subseteq$ cl int A. On the other hand, since A is closed, we have that $A \supseteq$ cl int A. Hence, A is regular. □

3. The distance to a downward set

Let A be a subset of $\mathbb{R}^I$ and $x \in \mathbb{R}^I$. We will use the following notations:

$$d_A(x) := \text{dist}\,(x, A) = \begin{cases} \inf\{\|x - a\| : a \in A\} & \text{if } A \neq \emptyset \\ +\infty & \text{if } A = \emptyset, \end{cases} \tag{3.1}$$

$$P_A(x) = \{a \in A : d_A(x) = \|x - a\|\}. \tag{3.2}$$

The function d_A is called a *distance function*. Geometrically, $P_A(x)$ is the set of all elements of A which are nearest to x . It is well-known that for each closed set $A \subseteq \mathbb{R}^I$ and each $x \in \mathbb{R}^I$ the set $P_A(x)$ is not empty, so the infimum in (3.1) is attained. It is well-known (and easy to check) that $P_A(x) \subseteq$bd A for each A and each $x \notin A$.

PROPOSITION 2. *Let A be a closed downward set and $x^0 \in \mathbb{R}^I$. Then there exists the least element $a^0 = \min P_A(x^0)$ of the set $P_A(x^0)$, and we have*

$$a^0 = x^0 - r\mathbf{1}, \tag{3.3}$$

where $r = d_A(x^0)$.

Proof. The result holds if $x^0 \in A$. Assume now that $x^0 \notin A$, that is, $r = d_A(x^0) > 0$, and let us consider the element a^0 defined by (3.3). We have $\|x^0 - a^0\| = r$ and for any $y \in B(x^0, r)$ there holds $y \geqslant x^0 - r\mathbf{1} = a^0$, so a^0 is the least element of the ball $B(x^0, r)$. Furthermore, since A is closed, $P_A(x^0) \neq \emptyset$. Let $a \in P_A(x^0)$. Then $\|x^0 - a\| = r$, that is, $a \in B(x^0, r)$. Since a^0 is the least element of $B(x^0, r)$, it follows that $a^0 \leqslant a$. Since the set A is downward, it follows that $a^0 \in A$. Hence $a^0 \in P_A(x^0)$ and a^0 is the least element of $P_A(x^0)$. □

COROLLARY 1. *Let A be a closed downward set, $x^0 \in \mathbb{R}^I$ and $a^0 = \min P_A(x^0)$. Then $a^0 \leqslant x^0$.*

Proof. By (3.3), we have $a^0 = x^0 - r\mathbf{1} \leqslant x^0$. □

COROLLARY 2. *The following is valid for a closed downward set A and any $x \in \mathbb{R}^I$:*

$$d_A(x) = \min\{\lambda \geqslant 0 : x - \lambda\mathbf{1} \in A\}. \tag{3.4}$$

Proof. If $x \in A$, then $x - 0 \cdot \mathbf{1} \in A$, so $\min\{\lambda \geqslant 0 : x - \lambda\mathbf{1} \in A\} = 0 = d_A(x)$. Let $x \notin A$. Then for any $\lambda > 0$ with $x - \lambda\mathbf{1} \in A$ we have

$$\lambda = \|\lambda\mathbf{1}\| = \|x - (x - \lambda\mathbf{1})\}\| \geqslant d_A(x),$$

and, by Proposition 2, $x - d_A(x)\mathbf{1} \in A$. Thus (3.4) holds. □

REMARK 1. An anonymous referee drew our attention to the following fact: Proposition 2 and Corollary 2 allow one to develop a simple numerical method for the search of the least element a^0 of the set $P_A(x^0)$. Indeed, by Proposition 2 and Corollary 2, this search can be reduced to the following one-dimensional optimization problem:

$$\lambda \to \min \quad \textit{subject} \quad \textit{to } x - \lambda\mathbf{1} \in A.$$

This problem can be solved for example by a well-known binary search method.

COROLLARY 3. *Let A be a closed downward set. Then*

$$\mathbb{R}^I \backslash A = \{a + \lambda\mathbf{1} : a \in \text{bd } A, \lambda > 0\}. \tag{3.5}$$

Proof. For any $x^0 \in \mathbb{R}^I \setminus A$, take $a = a^0$ of Proposition 2 (so $a \in \text{bd}\, A$) and $\lambda = r = d_A(x^0)$. Then, by (3.3), we obtain $x^0 = a + \lambda \mathbf{1}$.

Conversely, assume that there exist $a \in \text{bd}\ A$ and $\lambda > 0$ such that $a + \lambda \mathbf{1} \in A$. Let $V := \{x \in \mathbb{R}^I : x \ll a + \lambda \mathbf{1}\}$. Then V is an open neighbourhood of a. But, since $a + \lambda \mathbf{1} \in A$ and A is downward, it follows that $V \subseteq A$, so $a \notin \text{bd}\ A$, a contradiction. □

REMARK 2. (a) *If A is a non-empty downward subset of $\mathbb{R}^I$, then each $x \in \mathbb{R}^I$ satisfies*

$$\{\lambda \in \mathbb{R} : x \in \lambda \mathbf{1} + A\} \neq \emptyset. \tag{3.6}$$

Indeed, let $x \in \mathbb{R}^I, a \in A$. Choose any $\lambda \in \mathbb{R}$ such that $x - \lambda \mathbf{1} \leqslant a$ (e.g., $\lambda := \max_{i \in I}(x_i - a_i)$). Then, since A is downward, $x - \lambda \mathbf{1} \in A$, so $x \in \lambda \mathbf{1} + A$, which proves our assertion.

(b) Let A be a proper downward subset of $\mathbb{R}^I$ (i.e., such that $\emptyset \neq A \neq \mathbb{R}^I$), and $x \in \mathbb{R}^I$. For any $\lambda \in \mathbb{R}$ such that $x \in \lambda \mathbf{1} + A$, that is, $x - \lambda \mathbf{1} \in A$, and all $\lambda' \geqslant \lambda$ we have, since A is downward, $x - \lambda' \mathbf{1} \in A$, that is, $x \in \lambda' \mathbf{1} + A$. Thus, *the set* $\{\lambda \in \mathbb{R} : x \in \lambda \mathbf{1} + A\}$ *is a half-line in* $\mathbb{R}$, *either of the form* $(\rho, +\infty)$ *or of the form* $[\rho, +\infty)$. Also, clearly, *if* $A = \emptyset$, *then* $\{\lambda \in \mathbb{R} : x \in \lambda \mathbf{1} + A\} = \emptyset$, while *if* $A = \mathbb{R}^I$, *then* $\{\lambda \in \mathbb{R} : x \in \lambda \mathbf{1} + A\} = \mathbb{R}$.

(c) According to (a), *if A is a non-empty downward subset of $\mathbb{R}^I$, then*

$$\cup_{\lambda \in \mathbb{R}} (\lambda \mathbf{1} + A) = \mathbb{R}^I, \tag{3.7}$$

or, equivalently,

$$A + \cup_{\lambda \in \mathbb{R}} \lambda \mathbf{1} = \mathbb{R}^I; \tag{3.8}$$

by (b), in (3.7) and (3.8) one can replace $\lambda \in \mathbb{R}$ by $\lambda \in \mathbb{R}_+$.

DEFINITION 2. Let A be a downward set. The function $\rho_A : \mathbb{R}^I \to \bar{\mathbb{R}}$ defined by

$$\rho_A(x) = \inf\{\lambda \in \mathbb{R} : x \in \lambda \mathbf{1} + A\} \quad (x \in \mathbb{R}^I) \tag{3.9}$$

is called the *Minkowski gauge* with respect to addition (or *plus-Minkowski gauge*) of the set A.

An explanation of this term can be found in Section 10.

Note that *if A is a closed downward set,* then the inf in the definition of the plus-Minkowski gauge is attained, that is,

$$\rho_A(x) = \min\{\lambda \in \mathbb{R} : x \in \lambda \mathbf{1} + A\} \quad (x \in \mathbb{R}^I). \tag{3.10}$$

Let us give some simple examples.

PROPOSITION 3. *Let*

$$A^v = \{x \in \mathbb{R}^I : x \leqslant v\} \quad (v \in \mathbb{R}^I_+). \tag{3.11}$$

Then A^v is a closed downward set and

$$\rho_{A^v}(x) = \max_{i \in I}(x_i - v_i) \quad (x \in \mathbb{R}^I). \tag{3.12}$$

Proof. Let $x \in \mathbb{R}^I$. Then, since I is finite, we have

$$\begin{aligned}\rho_{A^v}(x) &= \inf\{\lambda \in \mathbb{R} : x - \lambda\mathbf{1} \leqslant v\} = \inf\{\lambda \in \mathbb{R} : x_i - \lambda \leqslant v_i \ (i \in I)\}\\ &= \inf\{\lambda \in \mathbb{R} : x_i - v_i \leqslant \lambda \ (i \in I)\} = \max_{i \in I}(x_i - v_i).\end{aligned}$$

□

PROPOSITION 4. *Let $l \in \mathbb{R}^I$ and*

$$D_l = \{x \in \mathbb{R}^I : \min_{i \in I}(x_i - l_i) \leqslant 0\}. \tag{3.13}$$

Then D_l is a closed downward set and

$$\rho_{D_l}(x) = \min_{i \in I}(x_i - l_i) \quad (x \in \mathbb{R}^I). \tag{3.14}$$

Proof. For any $x \in \mathbb{R}^I$ we have

$$\begin{aligned}\rho_{D_l}(x) &= \inf\{\lambda \in \mathbb{R} : x - \lambda\mathbf{1} \in D_l\} = \inf\{\lambda \in \mathbb{R} : \min_{i \in I}(x_i - \lambda - l_i) \leqslant 0\}\\ &= \inf\{\lambda \in \mathbb{R} : \min_{i \in I}(x_i - l_i) \leqslant \lambda\} = \min_{i \in I}(x_i - l_i).\end{aligned}$$

□

DEFINITION 3. Any set D_l of the form (3.13) will be called a *lower min-type half-space.*

REMARK 3. Geometrically, D_l is the complement of the 'open right angle' $\{x \in \mathbb{R}^I : x \gg l\}$.

THEOREM 1. *Let A be a closed downward set. Then $d_A = \rho_A^+$, that is,*

$$d_A(x) = \rho_A(x)^+ = \max(\rho_A(x), 0) \quad (x \in \mathbb{R}^I). \tag{3.15}$$

Proof. Let $x \in \mathbb{R}^I$. Then, by Corollary 2 and Remark 2, we have

$$d_A(x) = \min\{\lambda \geqslant 0 : x \in \lambda\mathbf{1} + A\} = \max\{\inf\{\lambda \in \mathbb{R} : x \in \lambda\mathbf{1} + A\}, 0\}. \tag{3.16}$$

But, by (3.9), the right hand side of (3.16) is just $\max(\rho_A(x), 0)$. □

REMARK 4. (a) From Proposition 1(b) it follows that for any downward set A we have

$$\text{int } A = \{x \in \mathbb{R}^I : \rho_A(x) < 0\}, \mathbb{R}^I \backslash \text{int } A = \{x \in \mathbb{R}^I : \rho_A(x) \geqslant 0\}. \tag{3.17}$$

On the other hand, clearly, $d_A(x) = 0$ for all $x \in \text{int } A$. Hence, by Theorem 1, for any closed downward set A we have

$$d_A(x) = \begin{cases} \rho_A(x) & \text{if } x \notin \text{int } A \\ 0 & \text{if } x \in \text{int } A. \end{cases} \tag{3.18}$$

(b) For any closed downward set A we have

$$\text{bd } A = \{x \in \mathbb{R}^I : \rho_A(x) = 0\}. \tag{3.19}$$

Indeed, if $x \in \text{bd } A = A \backslash \text{int } A$, then, by (3.10) and (3.17), we have $\rho_A(x) = 0$. Conversely, if $\rho_A(x) = 0$, then, by (3.17), we have $x \notin \text{int } A$, whence, by (3.18), $d_A(x) = \rho_A(x) = 0$, so $x \in A \backslash \text{int } A = \text{bd } A$.

PROPOSITION 5. *Let A be a closed downward set, $x \in \mathbb{R}^I$, and $\lambda > 0$.*
(a) *If $x \notin \text{int } A$, then*

$$d_A(x + \lambda\mathbf{1}) = d_A(x) + \lambda. \tag{3.20}$$

Hence:
(a1) *If $x \in \text{bd } A$ ($\subseteq A$, so $d_A(x) = 0$), then*

$$d_A(x + \lambda\mathbf{1}) = \lambda. \tag{3.21}$$

(a2) *If $x \notin A$ (so $d_A(x) > 0$), then*

$$d_A(x + \lambda\mathbf{1}) > \lambda.$$

(b) *If $x \in \text{int } A$, then*

$$d_A(x + \lambda\mathbf{1}) < \lambda. \tag{3.22}$$

Proof. (a) Assume that $x \notin \text{int } A$.
Case 1° : $x \in \text{bd } A$. Let $y = x + \lambda\mathbf{1}$ and let $\lambda' < \lambda$. Since $x \notin \text{int } A$, by Proposition 1 we have $x + (\lambda - \lambda')\mathbf{1} \notin A$, so

$$y - \lambda'\mathbf{1} = y - \lambda\mathbf{1} + (\lambda - \lambda')\mathbf{1} = x + (\lambda - \lambda')\mathbf{1} \notin A.$$

On the other hand, since $x \in \text{bd } A$ and A is closed, $y - \lambda\mathbf{1} = x \in A$. Hence, $\lambda = \min\{\lambda' : y - \lambda'\mathbf{1} \in A\}$ and therefore, by Corollary 2, $\lambda = d_A(y)$. Consequently,

$$d_A(x + \lambda\mathbf{1}) = d_A(y) = \lambda = d_A(x) + \lambda.$$

Case 2° : $x \notin A$. Let $r = d_A(x)$. Then, by Proposition 2, $x' := x - r\mathbf{1} \in$ bd A. Hence, applying (3.21), we obtain

$$d_A(x + \lambda\mathbf{1}) = d_A(x' + (r + \lambda)\mathbf{1}) = r + \lambda = d_A(x) + \lambda.$$

(b) Assume now that $x \in \operatorname{int} A$. Then, by (2.6), there exists an $\varepsilon > 0$ such that $x + \varepsilon\mathbf{1} \in A$. Hence

$$d_A(x + \lambda\mathbf{1}) \leqslant \|(x + \lambda\mathbf{1}) - (x + \varepsilon\mathbf{1})\| = \|(\lambda - \varepsilon)\mathbf{1}\| = \lambda - \varepsilon < \lambda.$$

□

4. Connections between downward sets, normal sets, and their approximation properties

We recall that a subset G of the cone $\mathbb{R}^I_+$ is called *normal* if $(x \in G,\ 0 \leqslant x' \leqslant x) \implies x' \in G$. For example, if f is an increasing function defined on $\mathbb{R}^I_+$, then its lower level sets $\{x \in \mathbb{R}^I_+ : f(x) \leqslant c\}$ for all c are normal.

Let G be a normal subset of $\mathbb{R}^I_+$. The intersection of all downward sets containing G is again downward. This set is called the *downward hull* of the set G. We shall denote the downward hull of a normal set G by G_*.

Let us indicate some properties of the downward hull of a normal set. For any set $X \subseteq \mathbb{R}^I$ we shall use the notation $X^+ = \{x^+ : x \in X\}$.

PROPOSITION 6. *Let $G_* \subseteq \mathbb{R}^I$ be the downward hull of a normal set $G \subseteq \mathbb{R}^I_+$. Then*

(1) $G_* = G - \mathbb{R}^I_+$.
(2) $G_* = \{x \in \mathbb{R}^I : x^+ \in G\}$.
(3) $G = G_* \cap \mathbb{R}^I_+$.
(4) G is closed if and only if G_* is closed.
(5) $(G_*)^+ = G$.

Proof. (1) By [14], p. 65, Proposition 2.3, we have

$$G_* = \{y \in \mathbb{R}^I : \exists g \in G, y \leqslant g\} = \{g - z : g \in G, z \in \mathbb{R}^I_+\} = G - \mathbb{R}^I_+. \tag{4.1}$$

(2) If $x \in G_*$, then, by (4.1), there exists $g \in G$ such that $g \geqslant x$. Since also $g \in G \subseteq \mathbb{R}^I_+$, we conclude that $g \geqslant \max(x, 0) = x^+$. Since G is normal, it follows that $x^+ \in G$.

Conversely, if $x^+ \in G$, then, since $x \leqslant x^+$, from (4.1) (with $g = x^+$) it follows that $x \in G_*$.

(3) The inclusion $G \subseteq G_* \cap \mathbb{R}^I_+$ is obvious. Conversely, if $x \in G_* \cap \mathbb{R}^I_+$, then, since $x \in \mathbb{R}^I_+$, we have $x = x^+$. Hence, by $x \in G_*$ and part (2), we obtain

$x = x^+ \in G$ (alternatively, by part (1), we have $0 \leqslant x = g - z \leqslant g$ for some $g \in G$ and $z \in \mathbb{R}^I_+$, and hence, since G is a normal set, $x \in G$).

(4) Assume now that G is closed, and let $g_k \in G_*$, $g_k \to g$. Then $g_k^+ \to g^+$. Also, by $g_k \in G_*$ and part (2), we have $g_k^+ \in G$. Hence, since G is closed, we obtain $g^+ \in G \subseteq G_*$. Consequently, since G_* is downward and $g \leqslant g^+$, it follows that $g \in G_*$. Thus, G_* is closed.

Conversely, if G_* is closed, then, by part (3), so is G.

(5) Let $x^+ \in (G_*)^+$, where $x \in G_*$. Then, by part (2), $x^+ \in G$.

Conversely, if $x \in G$, then, by part (3), $x \in G_*$ (so $x^+ \in (G_*)^+$) and $x = x^+$, whence $x \in (G_*)^+$. □

THEOREM 2. (a) *A downward set $A \subseteq \mathbb{R}^I$ is the downward hull of a normal set $G \subseteq \mathbb{R}^I_+$ if and only if*

$$A^+ \subseteq A. \tag{4.2}$$

(b) *A downward set $A \subseteq \mathbb{R}^I$ is the downward hull of a closed normal set $G \subseteq \mathbb{R}^I_+$ if and only if A is closed and satisfies* (4.2).

Moreover, in both (a) *and* (b) *the set G is unique, namely*

$$G = A \cap \mathbb{R}^I_+. \tag{4.3}$$

Proof. (a) If $A = G_*$, where $G \subseteq \mathbb{R}^I_+$ is a normal set, then, by Proposition 6, part (5), we have $A^+ = (G_*)^+ = G \subseteq G_* = A$.

Conversely, assume now that a downward set $A \subseteq \mathbb{R}^I$ satisfies (4.2), and define $G \subseteq \mathbb{R}^I_+$ by (4.3). Then, since A is a downward set, G is normal. Furthermore, if $a \in A$, then for $g := a^+$ we have $g \in A$ (by (4.2)) and $g \in \mathbb{R}^I_+$, so $g \in G$ (by (4.3)). Hence, since $a \leqslant g$, by the first equality in (4.1) we have $a \in G_*$. Thus, $A \subseteq G_*$. Conversely, by (4.3) we have $G \subseteq A$, whence, since A is downward, it follows that $G_* \subseteq A$. Thus, $A = G_*$.

(b) Assume that $A = G_*$, where $G \subseteq \mathbb{R}^I_+$ is a closed normal set. Then, by Proposition 6, part (4), A is closed. Also, by part (a), there holds (4.2).

Conversely, assume now that A is a closed downward set satisfying (4.2). Then the set $G \subseteq \mathbb{R}^I_+$ defined by (4.3) is closed and normal, and, by part (a), we have $A = G_*$.

Finally, observe that if a downward set A is the downward hull G_* of a closed normal set G, then, by Proposition 6, part (3), we have (4.3). This proves the uniqueness of G. □

We need also the following

PROPOSITION 7. *Let $x^0 \in \mathbb{R}^I_+$, $a^0 \in \mathbb{R}^I$. Then*

$$\|x^0 - a^0\| \geqslant \|x^0 - (a^0)^+\|. \tag{4.4}$$

Proof. Let

$$I_+ = \{i : a_i^0 > 0\},\ I_0 = \{i : a_i^0 = 0\}, I_- = \{i : a_i^0 < 0\}. \tag{4.5}$$

Then

$$(a_i^0)^+ = \begin{cases} a_i^0 & \text{if } i \in I_+ \cup I_0 \\ 0 & \text{if } i \in I_-, \end{cases} \tag{4.6}$$

and hence, since $x^0 \in \mathbb{R}_+^I$,

$$|x_i^0 - a_i^0| = |x_i^0 - (a_i^0)^+| \qquad (i \in I_+ \cup I_0),$$
$$|x_i^0 - a_i^0| = x_i^0 - a_i^0 > x_i^0 = |x_i^0 - (a_i^0)^+| \qquad (i \in I_-).$$

Thus $\|x^0 - a^0\| \geqslant \|x^0 - (a^0)^+\|$. □

COROLLARY 4. *Let $G \subseteq \mathbb{R}_+^I$ be a closed normal set, $G_* \subseteq \mathbb{R}^I$ the downward hull of G, and $x^0 \in \mathbb{R}_+^I$. Then*

$$d_{G_*}(x^0) = d_G(x^0). \tag{4.7}$$

Proof. Since $G_* \supseteq G$, we have $d_{G_*}(x^0) \leqslant d_G(x^0)$. On the other hand, by Proposition 6, part 2), and Proposition 7, for each $a \in G_*$ we have $a^+ \in G$ and $\|x^0 - a\| \geqslant \|x^0 - a^+\| \geqslant d_G(x^0)$, whence $d_{G_*}(x^0) \geqslant d_G(x^0)$. □

PROPOSITION 8. *Let G be a closed normal set and $x^0 \in \mathbb{R}_+^I$. Then there exists the least element g^0 of the set $P_G(x^0)$, namely $g^0 = (a^0)^+$, where a^0 is the least element of the set $P_{G_*}(x^0)$.*

Proof. By Proposition 2 and Corollary 4 we have $a^0 = x^0 - r\mathbf{1}$, where $r = d_{G_*}(x^0) = d_G(x^0)$. Hence $(a^0)^+ = \max(x^0 - r\mathbf{1}, 0)$, and thus, for the sets I_+, I_0 and I_- defined by (4.5) we obtain

$$((a^0)^+)_i = \begin{cases} x_i^0 - r & i \in I_+ \cup I_0 \\ 0 & i \in I_-. \end{cases} \tag{4.8}$$

It follows from Proposition 7 that $r = \|x^0 - a^0\| \geqslant \|x^0 - (a^0)^+\|$. Since $a^0 \in G_*$, we have $(a^0)^+ \in G$ (by Proposition 6, part 2)), whence $\|x^0 - (a^0)^+\| \geqslant r$. Thus $\|x^0 - (a^0)^+\| = r$, so $(a^0)^+ \in P_G(x^0)$. Let $g \in P_G(x^0)$. Then $\|g - x^0\| = r$. Since $g \in G \subseteq G_*$, it follows that $g \in P_{G_*}(x^0)$, so $g \geqslant \min P_{G_*}(x^0) = a^0$. We also have $g \geqslant 0$. Thus $g \geqslant (a^0)^+$, so $(a^0)^+ = \min P_G(x^0)$. □

REMARK 5. The existence of the least element g^0 of the set $P_G(x^0)$ and the formula $g^0 = (x^0 - r\mathbf{1})^+$ have been proved, with a different method, in [13]. Proposition 8 shows, in addition, that $g^0 = (a^0)^+$, the positive part of the least element of the set $P_{G_*}(x^0)$. Moreover, from Proposition 8 and Corollary 1 we obtain again the result of [13] that if G is a closed normal set, $x^0 \in \mathbb{R}_+^I \backslash G$ and g^0 is the least element of $P_G(x^0)$, then $g^0 \leqslant x^0$.

Now we shall extend formula (4.7) to an arbitrary element $x^0 \in \mathbb{R}^I$.

THEOREM 3. *Let $G \subseteq \mathbb{R}^I_+$ be a closed normal set, $G_* \subseteq \mathbb{R}^I$ the downward hull of G, and $x^0 \in \mathbb{R}^I$. Then*

$$d_{G_*}(x^0) = d_G((x^0)^+). \tag{4.9}$$

Proof. If $x^0 \in G_*$, then, by Proposition 6, part (2), $(x^0)^+ \in G$ and hence $d_{G_*}(x^0) = 0 = d_G((x^0)^+)$. If $x^0 \notin G_*$, let $r = d_{G_*}(x^0)$ (> 0) and let a^0 be the least element of the set $P_{G_*}(x^0)$. Then, by Proposition 2, $a^0 = x^0 - r\mathbf{1}$. Hence, by Proposition 6, part (2), $(x^0 - r\mathbf{1})^+ = (a^0)^+ \in G$. We claim that

$$(x^0)^+ - r\mathbf{1} \leqslant (x^0 - r\mathbf{1})^+. \tag{4.10}$$

Indeed, for each $i \in I$ we have

$$\begin{aligned} ((x^0)^+ - r\mathbf{1})_i &= ((x^0)^+)_i - r \\ &= (x_i^0)^+ - r = \max(x_i^0, 0) - r = \max(x_i^0 - r, -r) \\ &\leqslant \max(x_i^0 - r, 0) = (x_i^0 - r)^+ = (x^0 - r\mathbf{1})_i^+, \end{aligned}$$

which proves the claim (4.10). Hence, since G_* is downward, by $(x^0 - r\mathbf{1})^+ \in G \subseteq G_*$ and (4.10) we have

$$(x^0)^+ - r\mathbf{1} \in G_*. \tag{4.11}$$

Let (see Corollary 4)

$$d_{G_*}((x^0)^+) = d_G((x^0)^+) = r'. \tag{4.12}$$

Then, from Corollary 2 and (4.11) it follows that

$$r' = \min\{\lambda \geqslant 0 : (x^0)^+ - \lambda\mathbf{1} \in G_*\} \leqslant r. \tag{4.13}$$

In order to complete proof we need to show that $r' \geqslant r$. Let $g^0 \in G (\subseteq \mathbb{R}^I_+)$ be the least element of $P_G((x^0)^+)$. Then, by Remark 5, we have $g^0 \leqslant (x^0)^+$. Let

$$I_-^0 = \{i \in I : x_i^0 < 0\}. \tag{4.14}$$

Then for $i \in I_-^0$ we have $0 \leqslant g_i^0 \leqslant (x^0)_i^+ = 0$, so $g_i^0 = 0$. On the other hand, for $i \notin I_-^0$ we have

$$x_i^0 - g_i^0 = (x^0)_i^+ - g_i^0 \leqslant \|(x^0)^+ - g^0\| = d_G((x^0)^+) = r'. \tag{4.15}$$

Consider the vector a' defined by

$$a'_i = \begin{cases} x_i^0 & \text{if } i \in I_-^0 \\ g_i^0 & \text{if } i \notin I_-^0. \end{cases} \tag{4.16}$$

Then for $i \in I_-^0$ there holds $a'_i = x_i^0 < 0 \leqslant g_i^0$, and for $i \notin I_-^0$ we have $a'_i = g_i^0$, so $a' \leqslant g^0$, whence, since $g^0 \in G \subseteq G_*$, it follows that $a' \in G_*$. Furthermore, for $i \in I_-^0$ there holds $x_i^0 - a'_i = 0 = (x^0)_i^+ - g_i^0$, and for $i \notin I_-^0$ we have $x_i^0 - a'_i = (x^0)_i^+ - g_i^0$, so $x^0 - a' = (x^0)^+ - g^0$. Consequently,

$$r \leqslant \|x^0 - a'\| = \|(x^0)^+ - g^0\| = r'.$$

□

5. Lattices of downward sets, normal sets and distance functions

Let us first outline some properties of the family of all downward sets in $\mathbb{R}^I$.

(1) If A is a downward set in $\mathbb{R}^I$, then so is its closure cl A. Indeed, assume that A is non-empty and let $f \in \text{cl}\, A$ and $f' \leqslant f$. Let $f_n \in A$ be a sequence such that $f_n \to f$ and let $f'_n := \min(f', f_n)$ $(n = 1, 2, \ldots)$. Then, since $f'_n \leqslant f_n$ and A is a downward set, we have $f'_n \in A$ $(n = 1, 2, \ldots)$. Also, $f'_n \to \min(f', f) = f'$. Hence $f' \in \text{cl}\, A$.

(2) Let $(A_t)_{t\in T}$ be a family of downward sets in $\mathbb{R}^I$, where T is an arbitrary set of indices. Then both $\bigcup_{t\in T} A_t$ and $\bigcap_{t\in T} A_t$ are downward sets.

(3) If A is a downward set in $\mathbb{R}^I$, then so is the shift $x + A$, for each $x \in \mathbb{R}^I$ (the shift $x + \emptyset$ of the empty set is again empty, by definition).

Let us denote by $\mathcal{A}$ the family which consists of all closed downward sets in $\mathbb{R}^I$ (including the empty set $\emptyset$). Assume that $\mathcal{A}$ is equipped with the order relation by containment (i.e., $A_1 \leqslant A_2$ if and only if $A_1 \supseteq A_2$).

PROPOSITION 9. *The set $\mathcal{A}$ is a complete lattice. The supremum of a family $(A_t)_{t\in T}$ in $\mathcal{A}$ coincides with the intersection $\bigcap_{t\in T} A_t$; the infimum of this family in $\mathcal{A}$ coincides with the closure* cl $(\bigcup_{t\in T} A_t)$.

Proof. This follows from the properties (2) and (1) above. □

Let us consider the set $\mathcal{D}$ of all distance functions d_A, where $A \in \mathcal{A}$ (and where $d_\emptyset(x) = +\infty$ for all $x \in \mathbb{R}^I$). Assume that $\mathcal{D}$ is equipped with the pointwise order relation: $(d_1 \leqslant d_2) \Leftrightarrow (d_1(x) \leqslant d_2(x)$ for all $x \in \mathbb{R}^I)$. Let $\phi : \mathcal{A} \to \mathcal{D}$ be the mapping defined by

$$\phi(A) = d_A \quad (A \in \mathcal{A}). \tag{5.1}$$

PROPOSITION 10. *The mapping ϕ is an isomorphism between the ordered sets $\mathcal{A}$ and $\mathcal{D}$.*

Proof. First we show that ϕ is a one-to-one correspondence. Indeed, if $A_1 \neq A_2$, then the sets of zeroes of d_{A_1} and d_{A_2}, which are equal to A_1 and A_2 respectively, are different, so $d_{A_1} \neq d_{A_2}$. It follows from the definition that ϕ maps onto $\mathcal{D}$. Also, it is clear that ϕ is an increasing mapping: $d_{A_1}(x) \leqslant d_{A_2}(x)$ if $A_1 \supseteq A_2$ (that is, if $A_1 \leqslant A_2$). □

PROPOSITION 11. *The set $\mathcal{D}$ is a complete lattice.*

Proof. The result holds since ϕ is an order isomorphism between $\mathcal{A}$ and $\mathcal{D}$ and $\mathcal{A}$ is a complete lattice. □

We now calculate the infimum and the supremum in the lattice $\mathcal{D}$.

PROPOSITION 12. *Let T be a set of indices, $(A_t)_{t\in T}$ a family of closed downward sets, and $A = cl \bigcup_{t\in T} A_t (= \inf_{t\in T} A_t)$. Then*

$$d_A(x) = \inf_{t\in T} d_{A_t}(x) \qquad (x \in \mathbb{R}^I). \tag{5.2}$$

Proof. Let $A' = \bigcup_{t\in T} A_t$. By a well-known lemma (see, e.g., [7], lemma 2.1), we have

$$d_A(x) = \min_{a\in \mathrm{cl}\, A'} \|x - a\| = \inf_{a\in A'} \|x - a\| = \inf_{t\in T} \inf_{a\in A_t} \|x - a\| = \inf_{t\in T} d_{A_t}(x).$$

□

REMARK 6. We did not use in the proof that the sets A_t are closed and downward, nor the properties of the norm $\|.\|_\infty$.

COROLLARY 5. *Let $d_t \in \mathcal{D}$ $(t \in T)$. Then the infimum of the family d_t in the lattice $\mathcal{D}$ is the pointwise infimum.*

Proof. It follows from Proposition 12 that the pointwise infimum of a family of functions from $\mathcal{D}$ belongs to $\mathcal{D}$ as well. Hence, the conclusion follows (see e.g. [12], proof of Corollary 3.2). □

Let us recall the following definition [6]: Let X be a set of functions defined on a set Q. A set $Y \subset X$ is called an *infimal generator* of X if $x(q) = \inf\{y(q) : y \in Y, y \geqslant x\}$ for each $x \in X$ and $q \in Q$ (we assume that X is equipped with the natural pointwise order relation).

We can use the above to describe a small infimal generator of the set $\mathcal{D}$. For each $v \in \mathbb{R}^I$ let w_v be the function defined on $\mathbb{R}^I$ by

$$w_v(x) = (\max_i (x_i - v_i))^+ \qquad (x \in \mathbb{R}^I). \tag{5.3}$$

PROPOSITION 13. *The set $W := \{w_v : v \in \mathbb{R}^I\}$ is an infimal generator of the set $\mathcal{D}$.*

Proof. If $d \in \mathcal{D}$, then there exists a closed downward set A such that $d = d_A$. Since A is downward, we have $A = \bigcup_{v\in A} A^v$, where $A^v := \{x \in \mathbb{R}^I : x \leqslant v\}$. From (5.3), Proposition 3 and Theorem 1 it follows that

$$w_v(x) = \rho_{A^v}(x)^+ = d_{A^v}(x) \qquad (x \in \mathbb{R}^I), \tag{5.4}$$

so $w_v \in \mathcal{D}$. Also, by Proposition 12 and (5.4),

$$d(x) = d_A(x) = \inf_{v \in A} d_{A^v}(x) = \inf_{v \in A} w_v(x) \quad (x \in \mathbb{R}^I).$$

Thus W is an infimal generator of the set $\mathcal{D}$. □

Let us consider now the pointwise supremum of a family of functions from $\mathcal{D}$.

THEOREM 4. *Let T be a set of indices, $(A_t)_{t \in T}$ be a family of closed downward sets, and $A := \bigcap_{t \in T} A_t$. Then we have* (1.5)*, that is,*

$$d_A(x) = \sup_{t \in T} d_{A_t}(x) \quad (x \in \mathbb{R}^I). \tag{5.5}$$

Consequently, every family $\{A_t\}_{t \in T}$ of closed downward subsets of $\mathbb{R}^I$ is linearly regular.

Proof. Let $x \in \mathbb{R}^I$ and $r_t = d_{A_t}(x)$. Since $A \subseteq A_t$, we have $r_t \leqslant d_A(x)$ for all $t \in T$, whence $s := \sup_{t \in T} r_t \leqslant d_A(x)$. Thus, if $s = +\infty$, then $d_A(x) = +\infty$ and we have (5.5). Assume now that $s < +\infty$. Then the inequality $r_t \leqslant s$ implies $x - r_t\mathbf{1} \geqslant x - s\mathbf{1}$ for all $t \in T$. Hence, by $x - r_t\mathbf{1} \in A_t$ and since A_t is a downward set, we have $x - s\mathbf{1} \in A_t$, for each $t \in T$. Thus, $x - s\mathbf{1} \in \bigcap_{t \in T} A_t = A$ (so $A \neq \emptyset$), whence, by Corollary 2, $d_A(x) = \min\{\lambda \geqslant 0 : x - \lambda\mathbf{1} \in A\} \leqslant s$. Consider now any number $\lambda > 0$ such that $x - \lambda\mathbf{1} \in A$. Then $x - \lambda\mathbf{1} \in A_t$ $(t \in A)$, whence, by Corollary 2, $\lambda \geqslant d_{A_t}(x) = r_t$ $(t \in T)$, so $\lambda \geqslant \sup_{t \in T} r_t = s$. Applying again Corollary 2, we deduce that $d_A(x) = \min\{\lambda \geqslant 0 : x - \lambda\mathbf{1} \in A\} \geqslant s$. Consequently, $d_A(x) = s$, which proves (5.5). Hence (see Section 1), the family $\{A_t\}_{t \in T}$ is linearly regular. □

COROLLARY 6. *Let $d_t \in \mathcal{D}$ $(t \in T)$. Then the supremum of the family $(d_t)_{t \in T}$ in the lattice $\mathcal{D}$ is the pointwise supremum.*

Proof. Similar to that of Corollary 5. □

Let us also give an axiomatic characterization of distance functions to closed downward sets.

THEOREM 5. *Let d be a function defined on $\mathbb{R}^I$. Then $d \in \mathcal{D}$ if and only if either $d \equiv 0$ or the following properties hold:*

(1) $\min_{x \in \mathbb{R}^I} d(x) = 0$;
(2) *d is increasing and continuous*;
(3) *For each $y \in \mathbb{R}^I$ there exists $\lambda_y \in \mathbb{R}$ such that*

$$d(y + \lambda\mathbf{1}) = (\lambda - \lambda_y)^+ \quad (\lambda \in \mathbb{R}). \tag{5.6}$$

Proof. Let A be a closed downward set and $d := d_A$. If $A = \mathbb{R}^I$, then $d \equiv 0$. Assume now that $A \neq \mathbb{R}^I$. Then property (1) trivially holds and property (2) follows from Corollary 2. For any $y \in \mathbb{R}^I$, define

$$\lambda_y := \max\{\alpha \in \mathbb{R} : y + \alpha\mathbf{1} \in A\}. \tag{5.7}$$

Then, since $A \neq \mathbb{R}^I$ is a downward set, the max in (5.7) is attained and $\lambda_y < +\infty$. We have $y + \lambda_y\mathbf{1} \in A$ and $y + \lambda_y\mathbf{1} + \varepsilon\mathbf{1} \notin A$ for all $\varepsilon > 0$, whence, by Proposition 1(b), $y + \lambda_y\mathbf{1} \in \text{bd } A$. Hence, by Proposition 5(a) applied to $x = y + \lambda_y\mathbf{1}$, it follows that $d(y + \lambda_y\mathbf{1} + \mu\mathbf{1}) = \mu$ for all $\mu > 0$. Choose $\mu := (\lambda - \lambda_y)^+$. Then for $\lambda > \lambda_y$ we have $\mu = \lambda - \lambda_y > 0$, so $d(y + \lambda\mathbf{1}) = d(y + \lambda_y\mathbf{1} + \mu\mathbf{1}) = \mu = (\lambda - \lambda_y)^+$. On the other hand, by $y + \lambda_y\mathbf{1} \in A$ and since A is downward, for all $\lambda \in \mathbb{R}$ with $\lambda \leqslant \lambda_y$ we have $y + \lambda\mathbf{1} \in A$, whence $d(y + \lambda\mathbf{1}) = 0 = (\lambda - \lambda_y)^+$.

Conversely, if $d \equiv 0$ then $d = d_A$ for $A = \mathbb{R}^I$. Assume now that a function $d \not\equiv 0$ enjoys properties (1)-(3). Consider the closed downward set $A := \{x \in \mathbb{R}^I : d(x) = 0\}$. Then, by properties (1) and (2), we have $A \neq \emptyset$ and A is closed and downward. Also, clearly, $d(x) = 0 = d_A(x)$ for all $x \in A$. Let $x \in \mathbb{R}^I \backslash A$, $r = d_A(x)$ and $y = x - r\mathbf{1}$. Then, by Proposition 2, y is the least element of $P_A(x)$, whence $y \in \text{bd } A \subseteq A$. Hence, by (5.6) for $\lambda = 0$, it follows that $(-\lambda_y)^+ = d(y) = 0$, and so $\lambda_y \geqslant 0$. On the other hand, for all $\lambda > 0$ we have, by $y \in \text{bd } A$ and Proposition 1(b), $y + \lambda\mathbf{1} \notin A$, whence, by property (3) and the definition of A, we obtain $(\lambda - \lambda_y)^+ = d(y + \lambda\mathbf{1}) > 0$. This implies that $\lambda_y \leqslant 0$, and hence $\lambda_y = 0$. Thus, again by property (3), we can write $d(x) = d(y + r\mathbf{1}) = (r - \lambda_y)^+ = r$. □

Consider the family $\mathcal{G}$ of all closed normal subsets of the cone $\mathbb{R}^I_+$. Assume that $\mathcal{G}$ is equipped with the order relation by containment. Note that the intersection and the union of an arbitrary family of normal sets are again a normal set and the closure of a normal set is normal. As has been observed in [12], from this assertion it follows that the family $\mathcal{G}$ is a complete lattice: if $(G_t)_{t \in T}$ is a family of sets, $G_t \in \mathcal{G}$ for all $t \in T$, then

$$\sup_{t \in T} G_t = \bigcap_{t \in T} G_t, \quad \inf_{t \in T} G_t = \text{cl} \bigcup_{t \in T} G_t. \tag{5.8}$$

Consider the mapping $\psi : \mathcal{G} \to \mathcal{A}$, where

$$\psi(G) = G_*. \tag{5.9}$$

Note that, by the uniqueness part of Theorem 2, ψ is a one-to-one correspondence. Denote by $\mathcal{A}_0$ the image of the mapping ψ, that is, the family of all sets of the form $G - \mathbb{R}^I_+$ with $G \in \mathcal{G}$. Then $\mathcal{A}_0$ is a proper subset of $\mathcal{A}$ (see, e.g., Theorem 2(b)). Assume that $\mathcal{A}_0$ is ordered by containment. Clearly, ψ is an isomorphism of the ordered sets $\mathcal{G}$ and $\mathcal{A}_0$, hence $\mathcal{A}_0$ is a complete lattice. Also, for any family of normal sets $(G_t)_{t \in T}$ we have

$$\bigcup_{t \in T}(G_t - \mathbb{R}^I_+) = \bigcup_{t \in T}(G_t)_* = \left(\bigcup_{t \in T} G_t\right)_* = \left(\bigcup_{t \in T} G_t\right) - \mathbb{R}^I_+, \tag{5.10}$$

$$\bigcap_{t\in T}(G_t - \mathbb{R}^I_+) = \bigcap_{t\in T}(G_t)_* = \left(\bigcap_{t\in T} G_t\right)_* = \left(\bigcap_{t\in T} G_t\right) - \mathbb{R}^I_+. \tag{5.11}$$

Consequently, the infimum and the supremum of a family in the lattice $\mathcal{A}_0$ coincide with the infimum and supremum, respectively, of this family in the lattice $\mathcal{A}$.

From the above we can deduce the following result, which has been proved in [12] with a direct method:

PROPOSITION 14. ([12], Theorem 3.1). *Let $(G_t)_{t\in T}$ be a family of closed normal sets and $G = \bigcap_{t\in T} G_t$. Then*

$$d_G(x) = \sup_{t\in T} d_{G_t}(x) \qquad (x \in \mathbb{R}^I_+). \tag{5.12}$$

Proof. By formula (5.11), we have $G_* = \bigcap_{t\in T}(G_t)_*$. Hence, applying (4.7) and Theorem 4, we conclude that for $x \in \mathbb{R}^I_+$:

$$d_G(x) = d_{G_*}(x) = \sup_{t\in T} d_{(G_t)_*}(x) = \sup_{t\in T} d_{G_t}(x).$$

□

6. Duality: a multiplicative min-type coupling function and separation of a closed downward set and a ball

We shall consider the coupling function $\varphi_0 : \mathbb{R}^I \times \mathbb{R}^I_{++} \to \mathbb{R}$ defined by the 'scalar product'

$$\varphi_0(x, l) = \langle l, x\rangle := \min_{i\in I} l_i x_i \qquad (x \in \mathbb{R}^I, l \in \mathbb{R}^I_{++}). \tag{6.1}$$

Let us recall the following result of [13]:

THEOREM 6. ([13], Proposition 4.3 and its proof). *Let G be a closed normal set and $x^0 \in \mathbb{R}^I_+ \setminus G$. Assume that the least element g^0 of the set $P_G(x^0)$ is strictly positive. Then there exists a vector $l \in \mathbb{R}^I_{++}$ such that*

$$\langle l, g\rangle \leqslant 1 \leqslant \langle l, y\rangle \qquad (g \in G, y \in B_0(x^0, r)), \tag{6.2}$$

where $r = d_G(x)$ and $B_0(x^0, r) = \{y \in \mathbb{R}^I_+ : \|x^0 - y\| \leqslant r\}$.

We shall apply this result to prove the following theorem.

THEOREM 7. *For a set $A \subseteq \mathbb{R}^I$, the following statements are equivalent:*
1°. A is downward and closed.

2°. For each $x^0 \notin A$ there exist a strictly positive vector l and a number $\lambda > 0$ such that

$$\langle l, a + \lambda\mathbf{1}\rangle \leqslant 1 \leqslant \langle l, y + \lambda\mathbf{1}\rangle \quad (a \in A,\ \ y \in B(x^0, r)), \tag{6.3}$$

where $r = d_A(x^0)$ and $B(x^0, r) = \{y \in \mathbb{R}^I : \|x^0 - y\| \leqslant r\}$.

3°. For each $x^0 \notin A$ there exist a strictly positive vector l and numbers $\lambda, r > 0$ such that

$$\langle l, a + \lambda\mathbf{1}\rangle \leqslant \langle l, y + \lambda\mathbf{1}\rangle \quad (a \in A,\ \ y \in B(x^0, r)). \tag{6.4}$$

Proof. $1° \Rightarrow 2°$. Assume 1° and let $a^0 = x^0 - r\mathbf{1}$ be the least element of the set $P_A(x^0)$ (see Proposition 2). Let $\lambda > r$. Consider the set $A^\lambda := A + \lambda\mathbf{1}$ and the element $x^\lambda = x^0 + \lambda\mathbf{1}$. Then $x^\lambda \in \mathbb{R}^I_+$ for sufficiently large λ, and we have

$$d_{A^\lambda}(x^\lambda) = \min_{a\in A} \|(x^0 + \lambda\mathbf{1}) - (a + \lambda\mathbf{1})\| = \min_{a\in A} \|x^0 - a\| = d_A(x) = r.$$

Let a^λ be the least element of the set $P_{A^\lambda}(x^\lambda)$. Then a^λ coincides with the shift $a^0 + \lambda\mathbf{1}$ of the least element a^0 of $P_A(x)$, and hence the inequality $a^\lambda \gg 0$ holds for sufficiently large λ. Therefore, taking such a λ, and applying Theorem 6 to the closed normal set $G := A^\lambda \cap \mathbb{R}^I_+$, it follows that there exists a vector $l \gg 0$ such that

$$\langle l, a'\rangle \leqslant 1 \leqslant \langle l, y'\rangle \quad (a' \in A^\lambda \cap \mathbb{R}^I_+,\ \ y' \in B_0(x^0 + \lambda\mathbf{1}, r)), \tag{6.5}$$

whence also

$$\langle l, a'\rangle \leqslant 1 \leqslant \langle l, y'\rangle \quad (a' \in A^\lambda,\ \ y' \in B_0(x^0 + \lambda\mathbf{1}, r)), \tag{6.6}$$

since for each $a' \in \mathbb{R}^I \backslash \mathbb{R}^I_+$ we have $\langle l, a'\rangle < 0$ (by $l \gg 0$ and (6.1)).

Consider the ball $B(x^0 + \lambda\mathbf{1}, r)$. Since

$$B(x^0 + \lambda\mathbf{1}, r) = \{y' :\ x^0 + \lambda\mathbf{1} - r\mathbf{1} \leqslant y' \leqslant x^0 + \lambda\mathbf{1} + r\mathbf{1}\}, \tag{6.7}$$

we have $B(x^0 + \lambda\mathbf{1}, r) \subset \mathbb{R}^I_{++}$ for sufficiently large λ, so $B(x^0 + \lambda\mathbf{1}, r) = B_0(x^0 + \lambda\mathbf{1}, r)$. Also, clearly, $y' \in B(x^0 + \lambda\mathbf{1}, r)$ if and only of $y' - \lambda\mathbf{1} \in B(x^0, r)$. Thus, (6.6) is equivalent to (6.3).

The implication $2° \Rightarrow 3°$ is obvious.

$3° \Rightarrow 1°$. Assume 3°. If A is not downward, then there exist $a \in A$ and $x^0 \in \mathbb{R}^I \backslash A$ such that $x^0 \leqslant a$. Then, by 3°, there exist a strictly positive vector l and numbers $\lambda, r > 0$ satisfying (6.4). Hence, for $y = x^0 - r\mathbf{1} \in B(x^0, r)$ (so $y \leqslant a - r\mathbf{1}$) we obtain

$$\langle l, a + \lambda\mathbf{1}\rangle \leqslant \langle l, x^0 - r\mathbf{1} + \lambda\mathbf{1}\rangle \leqslant \langle l, a - r\mathbf{1} + \lambda\mathbf{1}\rangle,$$

which is impossible. Thus, A is downward.

Finally, if A is not closed, then there exists a sequence $\{x^k\} \subseteq A$ converging to some $x^0 \in \mathbb{R}^I \backslash A$. Then, by 3°, there exist a strictly positive vector l and numbers $\lambda, r > 0$ satisfying (6.4). Hence, for $a = x^k$ with k large enough so as to have $x^k \gg x^0 - r\mathbf{1}$, and $y = x^0 - r\mathbf{1} \in B(x^0, r)$, we obtain

$$\langle l, x^0 - r\mathbf{1} + \lambda\mathbf{1}\rangle < \langle l, x^k + \lambda\mathbf{1}\rangle \leqslant \langle l, x^0 - r\mathbf{1} + \lambda\mathbf{1}\rangle,$$

which is impossible. Thus, A is closed. □

7. Duality: an additive min-type coupling function and separation of a closed downward set and a ball

DEFINITION 4. Let $x \in \mathbb{R}^I$. The set

$$\{x + \lambda\mathbf{1} \in \mathbb{R}^I : \lambda \in R\} \tag{7.8}$$

is called the *diagonal line* passing through x.

DEFINITION 5. A set $U \subseteq \mathbb{R}^I$ is said to be *closed along diagonal lines*, if

$$(x \in \mathbb{R}^I, \lambda_k \in \mathbb{R}, x + \lambda_k\mathbf{1} \in U, k = 1, 2, ..., \lambda_k \to \lambda) \Rightarrow x + \lambda\mathbf{1} \in U. \tag{7.9}$$

We shall consider the coupling function $\varphi : \mathbb{R}^I \times \mathbb{R}^I \to \mathbb{R}$ defined by

$$\varphi(x, l) := \min_{i \in I} (l_i + x_i) \quad (x \in \mathbb{R}^I, l \in \mathbb{R}^I). \tag{7.10}$$

THEOREM 8. *For a subset A of $\mathbb{R}^I$ the following statements are equivalent:*
1°. A is a closed downward set.
2°. A is closed along diagonal lines and downward.
3°. For each $x \in \mathbb{R}^I \backslash A$ there exists $l \in \mathbb{R}^I$ such that

$$\sup_{a \in A} \varphi(a, l) \leqslant 0 < \varphi(x, l). \tag{7.11}$$

4°. For each $x \in \mathbb{R}^I \backslash A$ there exists $l \in \mathbb{R}^I$ such that

$$\sup_{a \in A} \varphi(a, l) < \varphi(x, l). \tag{7.12}$$

Proof. The implication 1° ⇒ 2° is obvious.

2° ⇒ 3°. Assume that A is closed along diagonal lines and downward and let $x \in \mathbb{R}^I \backslash A$. Then, since A is closed along diagonal lines, there exists $\lambda > 0$ such that $x - \lambda\mathbf{1} \notin A$ (indeed, if $x - \lambda\mathbf{1} \in A$ for all $\lambda > 0$, then, taking $\lambda \searrow 0$, we obtain $x \in A$, in contradiction with our assumption). Define $l := (l_i)_{i \in I} \in \mathbb{R}^I$ by $l := \lambda\mathbf{1} - x$, i.e.,

$$l_i := \lambda - x_i \quad (i \in I). \tag{7.13}$$

Let $a \in A$. Then

$$\varphi(a, l) = \min_{i \in I} (a_i + l_i) < 0; \tag{7.14}$$

indeed, otherwise $\min_{i \in I} (a_i + l_i) \geqslant 0$, whence $a_i \geqslant -l_i = x_i - \lambda$ $(i \in I)$, and hence, since A is a downward set, $x - \lambda \mathbf{1} \in A$, in contradiction with the choice of λ. On the other hand,

$$\varphi(x, l) = \min_{i \in I} (x_i + l_i) = \min_{i \in I} (x_i + (\lambda - x_i)) = \lambda > 0, \tag{7.15}$$

which, together with (7.14), yields (7.11).

The implication 3° ⇒ 4° is obvious.

4° ⇒ 1°. Assume that A is a set with the separation property (7.12), and that there exist $a^0 \in A$ and $x \leqslant a^0$ with $x \notin A$. Then, by the separation assumption, there exists $l \in \mathbb{R}^I$ such that

$$\min_{i \in I} (a_i^0 + l_i) \leqslant \sup_{a \in A} \min_{i \in I} (a_i + l_i) < \min_{i \in I} (x_i + l_i),$$

and hence there exists an index $j \in I$ such that $a_j^0 + l_j < x_j + l_j$, in contradiction with the assumption $x \leqslant a^0$. Consequently, A is a downward set.

Finally, assume again that A is a set with the separation property (7.12), and assume that there exist $a^k \in A$ $(k = 1, 2, \ldots)$, $a^k \to x \in \mathbb{R}^I \setminus A$. Then, by the separation assumption, there exists $l \in \mathbb{R}^I$ such that

$$\varphi(a^k, l) \leqslant \sup_{a \in A} \varphi(a, l) := M < \varphi(x, l) \quad (k = 1, 2, \ldots). \tag{7.16}$$

But, since the set $\{y \in \mathbb{R}^I : \varphi(y, l) \leqslant M\}$ is closed (by the continuity of $\varphi(., l)$), from $a^k \to x$ and the first part of (7.16) it follows that $\varphi(x, l) \leqslant M$, which contradicts the second part of (7.16). Consequently, A is a closed set. □

REMARK 7. We recall (see [14, 11]) that if X is a set and W is a set of functions $w : X \to \bar{\mathbb{R}}$, a subset A of X is said to be *abstract convex with respect to* W, or, briefly, W*-convex*, if for each $x \in X \setminus A$ there exists $w \in W$ such that $\sup w(A) < w(x)$. By the above Theorem, *a set* $A \subseteq \mathbb{R}^I (= X)$ *is closed and downward if and only if it is closed along diagonal lines and downward, or, if and only if it is* $\widetilde{L}$*-convex, where* $\widetilde{L}(= W)$ denotes the set of all functions $\varphi(., l), l \in \mathbb{R}^I$.

Let

$$\|l\|_* := \sup_{\|x\|_\infty \leqslant 1} \varphi(x, l). \tag{7.17}$$

The following proposition gives the explicit expression for $\|l\|_*$.

PROPOSITION 15. *We have*

$$\|l\|_* = 1 + \min_{i \in I} l_i \quad (l \in \mathbb{R}^I), \tag{7.18}$$

$$\|-l\|_* = 1 - \max_{i \in I} l_i. \quad (l \in \mathbb{R}^I). \tag{7.19}$$

Thus, in general, $\|l\|_* \neq \|-l\|_*$.

Proof. By (7.17), we have

$$\|l\|_* = \sup_{\|x\|_\infty \leqslant 1} \varphi(x, l) = \sup_{\|x\|_\infty \leqslant 1} \min_{i \in I} (l_i + x_i) = 1 + \min_{i \in I} l_i,$$

which proves (7.18). This, in turn, yields (7.19). □

8. Characterization of best approximations by a separation property

In the sequel we shall use the following simple lemma.

LEMMA 1. *Let A be a downward set and $a^0 \in \text{bd}\, A$. Then*

$$\varphi(a, -a^0) = \min_{i \in I} (a_i - a_i^0) \leqslant 0 \quad (a \in A). \tag{8.1}$$

Proof. Assume, on contrary, that there exists $a \in A$ such that $a_i^0 < a_i$ $(i \in I)$. Then the set $V := \{x \in \mathbb{R}^I \,|\, x \ll a\}$ is a neighbourhood of a^0, and, since A is a downward set, in V there exists no element $x \notin A$, in contradiction with the assumption that $a^0 \in \text{bd}\, A$. □

THEOREM 9. *Let A be a closed downward set, $x^0 \in \mathbb{R}^I \backslash A$, $a' \in A$ and $r' := \left\|x^0 - a'\right\|$. We have $a' \in P_A(x^0)$ if and only if there exists $l \in \mathbb{R}^I$ such that*

$$\varphi(a, l) \leqslant 0 \leqslant \varphi(y, l) \quad (a \in A,\, y \in B(x^0, r')), \tag{8.2}$$

Moreover, if (8.2) *holds with $l = -a'$, then $a' = a^0 := \min P_A(x^0)$.*

Proof. Necessity. Assume that $a' \in P_A(x^0)$, so $r' = d_A(x^0) := r$, and let us show that there exists $l \in \mathbb{R}^I$ such that

$$\varphi(a, l) \leqslant 0 \leqslant \varphi(y, l) \quad (a \in A,\, y \in B(x^0, r)). \tag{8.3}$$

Define $l := -a^0 = -\min P_A(x^0)$, and let $y \in B(x^0, r)$, that is,

$$-r \leqslant y_i - x_i^0 \leqslant r \quad (i \in I). \tag{8.4}$$

Then, by $-l = a^0 = x^0 - r\mathbf{1}$ (see Proposition 2) and (8.4), we obtain

$$\varphi(y, l) = \min_{i \in I} (y_i + l_i) = \min_{i \in I} (y_i - (x_i^0 - r)) \geqslant 0,$$

i.e., the right hand side of (8.3). On the other hand, the left hand side of (8.3) for $l = -a^0$ holds by lemma 1.

Sufficiency. Assume that there exists $l \in \mathbb{R}^I$ satisfying (8.2). Since $x^0 - r'\mathbf{1} \in B(x^0, r')$, by (8.2) we have $\min_{i\in I}(x_i^0 - r' + l_i) = \varphi(x^0 - r'\mathbf{1}, l) \geqslant 0$, whence

$$l_i \geqslant -x_i^0 + r' \quad (i \in I). \tag{8.5}$$

Now let $a \in A$ be arbitrary, and define $a'' \in \mathbb{R}^I$ by $a'' := \min(a, x^0)$, i.e.,

$$a_i'' = \begin{cases} a_i & \text{if } a_i \leqslant x_i^0 \\ x_i^0 & \text{if } a_i > x_i^0. \end{cases} \tag{8.6}$$

Then, by (8.2) and (8.5), we have

$$0 \geqslant \varphi(a'', l) = \min_{i\in I}(a_i'' + l_i) \geqslant \min_{i\in I}(a_i'' - x_i^0 + r') = r' + \min_{i\in I}(a_i'' - x_i^0),$$

whence $\min_{i\in I}(a_i'' - x_i^0) \leqslant -r'$, and hence

$$\|x^0 - a''\| = \max_{i\in I}|x_i^0 - a_i''| \geqslant r' = \|x^0 - a'\|. \tag{8.7}$$

Finally, let $I^0 := \{i \in I \,|\, a_i \leqslant x_i^0\}$. Then

$$\|x^0 - a\| = \max_{i\in I}|x_i^0 - a_i| \geqslant \max_{i\in I^0}|x_i^0 - a_i| = \max_{i\in I}|x_i^0 - a_i''| = \|x^0 - a''\|,$$

which, together with (8.7), yields $\|x^0 - a\| \geqslant \|x^0 - a'\|$. Thus, $a' \in P_A(x^0)$.

In order to prove the last statement, assume that (8.2) holds with $l = -a'$, i.e.,

$$\varphi(a, -a') \leqslant 0 \leqslant \varphi(y, -a') \quad (a \in A,\ y \in B(x^0, r')). \tag{8.8}$$

Then, by the above, $a' \in P_A(x^0)$. On the other hand, for any $a \in P_A(x^0)$ we have $\|x^0 - a\| = \|x^0 - a'\| = r'$, so $a \in B(x^0, r')$, whence, by (8.8) (for $y = a$), $\min_{i\in I}(a_i - a_i') = \varphi(a, -a') \geqslant 0$. Thus, $a \geqslant a'$, whence, by $a' \in P_A(x^0)$ and the definition of a^0, we obtain $a' = a^0$. □

9. The distance to a downward set revisited

THEOREM 10. *Let A be a closed downward set and $x^0 \in \mathbb{R}^I \setminus A$. Then*

$$d_A(x^0) = \max_{\substack{D\in\mathcal{H}\\ A\subseteq D}} d_D(x^0), \tag{9.1}$$

where $\mathcal{H}$ denotes the set of all lower min-type half-spaces (see Definition 3).

Proof. Clearly, we have (for any set A)

$$d_A(x^0) \geqslant \sup_{\substack{D\in\mathcal{H}\\ A\subseteq D}} d_D(x^0). \tag{9.2}$$

We shall show that there exists $D \in \mathcal{H}$ such that $A \subseteq D$ and $d_A(x^0) \leqslant d_D(x^0)$, which will complete the proof.

Let $a^0 := \min P_A(x^0)$ and define $D \in \mathcal{H}$ by

$$D := \{x \in \mathbb{R}^I \,|\, \varphi(x, -a^0) \leqslant 0\} = \{x \in \mathbb{R}^I : \min_{i\in I} (x_i - a_i^0) \leqslant 0\}. \tag{9.3}$$

Let $a \in A$. We claim that $a \in D$. Indeed, assume, on contary, that $a \notin D$, i.e., $a_i > a_i^0$ for all $i \in I$. Define

$$\overline{a}_i := \min(a_i, x_i^0) \quad (i \in I). \tag{9.4}$$

Then $\bar{a} \leqslant a$, whence, since A is downward, $\bar{a} \in A$. Furthermore, since $x_i^0 - a_i^0 = d_A(x^0) > 0$, whence $x_i^0 > a_i^0$, for all $i \in I$ (by Proposition 2), we have $\bar{a}_i > a_i^0$ for all $i \in I$. Hence, $\left\|x^0 - \bar{a}\right\| = \max_{i\in I}(x_i^0 - \bar{a}_i) < \max_{i\in I}(x_i^0 - a_i^0) = \left\|x^0 - a^0\right\|$, in contradiction with $a^0 \in P_A(x^0)$. This proves the claim $a \in D$. Thus, $A \subseteq D$.

Finally, since D is closed, there exists $x \in D$ such that $d_D(x^0) = \left\|x^0 - x\right\|$. Then, since $x \in D$ of (9.3), there exists $j \in I$ such that $x_j \leqslant a_j^0$ and hence

$$d_D(x^0) = \left\|x^0 - x\right\| \geqslant x_j^0 - x_j \geqslant x_j^0 - a_j^0 = d_A(x^0)\,,$$

where the last equality holds by $a^0 = \min P_A(x^0)$ (see Proposition 2). □

REMARK 8. Since $\|.\| = \|.\|_\infty$ and I is finite, we have

$$d_A(x^0) = \min_{a\in A} \max_{i\in I} |x_i^0 - a_i^0|.$$

On the other hand, by Proposition 4 and Theorem 1, there holds

$$\max_{\substack{D\in\mathcal{H}\\ A\subseteq D}} d_D(x^0) = \max_{\substack{l\in\mathbb{R}^I\\ A\subseteq D_l}} d_{D_l}(x^0) = \max_{\substack{l\in\mathbb{R}^I\\ \min_{i\in I}(a_i - l_i)\leqslant 0\ (a\in A)}} \min_{i\in I} (x_i - l_i)^+.$$

Consequently, we can write (9.1) as the following *duality theorem*:

$$\min_{a\in A} \max_{i\in I} |x_i^0 - a_i^0| = \max_{\substack{l\in\mathbb{R}^I\\ \min_{i\in I}(a_i - l_i)\leqslant 0\ (a\in A)}} \min_{i\in I} (x_i - l_i)^+. \tag{9.5}$$

10. Connections between the multiplicative and additive cases

In this section we shall show that the multiplicative and additive cases above are closely related, in two ways.

(1) We shall adopt the usual notations

$$e^x = (e^{x_i})_{i\in I} \quad (x = (x_i)_{i\in I} \in \mathbb{R}^I), \tag{10.1}$$

$$\ln y = (\ln y_i)_{i\in I} \quad (y = (y_i)_{i\in I} \in \mathbb{R}^I_{++}). \tag{10.2}$$

If G is a normal subset of $\mathbb{R}^I_{++}$, then the set A defined by

$$A := \ln G = \{\ln g : g \in G\} \tag{10.3}$$

is a downward subset of $\mathbb{R}^I$ and each downward set A can be represented in the form (10.3), taking $G := e^A = \{e^a : a \in A\}$. Note also that G is closed if and only if $\ln G$ is closed.

One has also a correspondence between functions, as follows. Let us denote by P the set of all functions $p : \mathbb{R}^I_{++} \to \bar{\mathbb{R}}_+$ and by Q the set of all functions $h : \bar{\mathbb{R}}^I \to \bar{\mathbb{R}}$. Let us consider the mapping $V : P \to Q$ defined by

$$V(p)(x) := \ln p(e^x) \quad (p \in P, x \in \mathbb{R}^I). \tag{10.4}$$

It is well-known (see e.g. [4]) that V is one-to-one, namely the inverse mapping V^{-1} is

$$V^{-1}(h)(y) = e^{h(\ln y)} \quad (h \in Q, y \in \mathbb{R}^I_{++}). \tag{10.5}$$

We recall that if G is a subset of $\mathbb{R}^I_{++}$, the usual Minkowski gauge of G is the function $\mu_G : \mathbb{R}^I_{++} \to \mathbb{R}_+$ defined by

$$\mu_G(y) := \inf\{\lambda > 0 : y \in \lambda G\} \quad (y \in R^I_{++}). \tag{10.6}$$

PROPOSITION 16. *Let $G \subseteq \mathbb{R}^I_{++}$ be a normal set and let $A := \ln G$. Then*

$$V(\mu_G) = \rho_A, \tag{10.7}$$

where ρ_A is the plus-Minkowski gauge (3.9) *of A.*

Proof. Let $x \in \mathbb{R}^I$. Then, by (10.6),

$$\begin{aligned}\mu_G(e^x) &= \inf\{\lambda > 0 : e^x \in \lambda G\} = \inf\{\lambda > 0 : x \in (\ln\lambda)\mathbf{1} + \ln G\}\\ &= \inf\{\lambda > 0 : x \in (\ln\lambda)\mathbf{1} + A\},\end{aligned} \tag{10.8}$$

and hence, by (10.4), (10.8), and (3.9),

$$\begin{aligned}V(\mu_G)(x) &= \ln \mu_G(e^x) = \inf\{\ln\lambda : \lambda > 0, x \in \ln\lambda + A\}\\ &= \inf\{\nu : x \in \nu\mathbf{1} + A\} = \rho_A(x).\end{aligned}$$

□

PROPOSITION 17. *Let* $\widetilde{\varphi}_0$ *be the restriction of the multiplicative coupling function* (6.1) *to* $\mathbb{R}^I_{++} \times \mathbb{R}^I_{++}$, *that is,*

$$\widetilde{\varphi}_0(x, l) = \varphi_0(x, l) = \min_{i \in I} l_i x_i \qquad (x, l \in \mathbb{R}^I_{++}), \tag{10.9}$$

and let φ *be the additive coupling function* (7.10). *Then* $\widetilde{\varphi}_0(., l) \in P$ *and*

$$V(\widetilde{\varphi}_0(., l))(x) = \varphi(x, \ln l) \qquad (x \in \mathbb{R}^I, l \in \mathbb{R}^I_{++}). \tag{10.10}$$

Proof. Let $x \in \mathbb{R}^I$, $l \in \mathbb{R}^I_{++}$. Then, by (10.4) (with $p = \widetilde{\varphi}_0(., l)$ of (10.9)), (6.1), and (7.10), we obtain

$$\begin{aligned} V(\widetilde{\varphi}_0(., l))(x) &= \ln \varphi_0(e^x, l) = \ln \min_{i \in I} l_i e^{x_i} = \min_{i \in I} \ln l_i e^{x_i} \\ &= \min_{i \in I}(x_i + \ln l_i) = \varphi(x, \ln l). \end{aligned}$$

□

Thus, the mappings (10.3) and (10.4) permit to carry results on subsets of $\mathbb{R}^I_{++}$ and extended-non-negative valued functions on $\mathbb{R}^I_{++}$ to corresponding results on subsets of $\mathbb{R}^I$ and extended-real valued functions on $\mathbb{R}^I$ respectively, and vice versa.

(2) Many results from the additive and the multiplicative cases can be extended to the following framework:

DEFINITION 6. Let I be a non-empty finite index set. Let $A = (A, \diamond, \leqslant)$ be a complete ordered group, where $\diamond$ denotes the group operation (we recall that a partially oredered group $A = (A, \diamond, \leqslant)$ is said to be complete, if $(A, \leqslant)$ is a conditionally complete lattice, i.e., if every non-empty order-bounded subset of A admits a supremum and an infimum in A). Define a coupling function $\varphi : A^I \times A^I \to A$ by

$$\varphi(x, l) := \inf_{i \in I} (x_i \diamond l_i) \qquad (x, l \in A^I). \tag{10.11}$$

Then, for $A = (\mathbb{R}_{++}, \times, \leqslant)$, φ becomes the coupling function $\widetilde{\varphi}_0 : \mathbb{R}^I_{++} \times \mathbb{R}^I_{++} \to \mathbb{R}_{++}$ defined by (10.9), and for $A = (\mathbb{R}, +, \leqslant)$, φ becomes the coupling function $\mathbb{R}^I \times \mathbb{R}^I \to \mathbb{R}$ defined by (7.10).

The generalization of some results of this paper to the framework of functions with values in extensions of complete ordered groups [8, 9] will be given elsewhere (in preparation).

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